Magic generation in the dual-unitary XXZ model

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Alexander von HUMBOLDT STIFTUNG

Introduction

My goal is understanding many-body quantum **dynamics** and resulting quantum **states**

The states have special quantum properties, e.g.

- Entanglement
- Nonstabilizerness, also known as magic

Together with

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Jordi Arnau Montañà López
J. Phys. A: Math. Theor. 57 475301 (2024)

Neil Dowling

Xhek Turkesh

arXiv:2408.16047

Quantum quench

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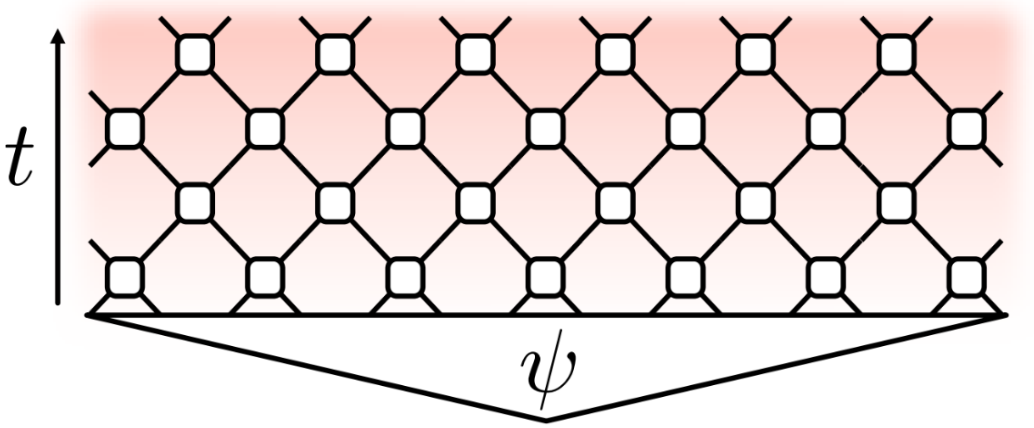
Interested in magic of states, relevant for dynamics of many-body systems

Start with some **simple initial state**, e.g.

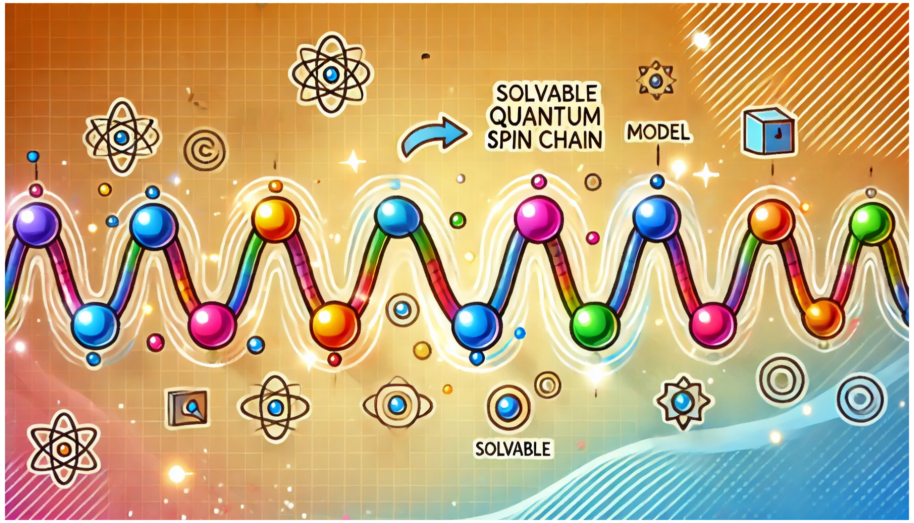
$$|\psi(0)\rangle = |\phi^+\rangle^{\otimes N/2} \qquad |\phi^+\rangle = \frac{1}{\sqrt{2}}\sum_{i=0}^{1}|ii\rangle$$

Time evolve them with H or a **quantum circuit**



How is magic generated? What are its properties?

Particular model: Dual unitary XXZ



Study magic with **solvable toy models**

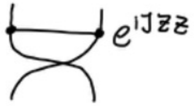
Many exact results for dual-unitary models

B Bertini, PK, T Prosen, PRL 2019

Not enough, so we need to restrict more

- Dual-unitary XXZ, with a local gate

$$U_{\mathrm{e,o}} = \exp\left(-\mathrm{i}J_{\mathrm{e,o}}\sigma_z \otimes \sigma_z\right) \cdot \mathrm{SWAP} =$$



Stabilizer Rényi entropy (SRE)

L. Leone, S. F. E. Oliviero, and A. Hamma, PRL 128, 050402 (2022)

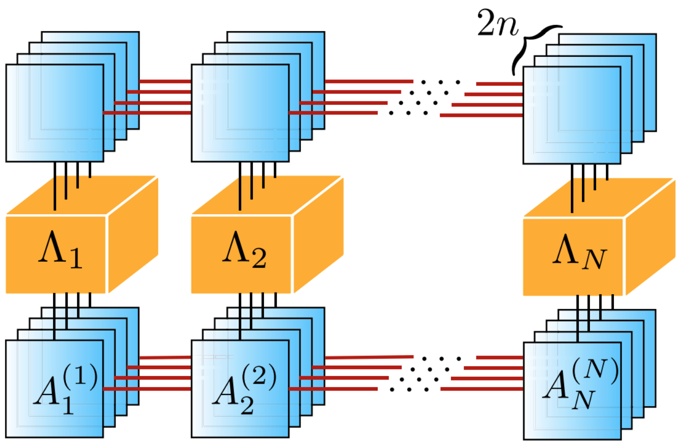
Magic monotone

$$\zeta_n(|\psi\rangle) := \frac{1}{2^N} \sum_{P \in \mathcal{P}_N} \langle\psi|P|\psi\rangle^{2n}$$

$$M_n(|\psi\rangle) = \frac{1}{1-n} \log\left(\zeta_n(|\psi\rangle)\right)$$

Nice to work with, as it has a tensor network expression

T. Haug and L. Piroli, PRB 107 5148 (2023)



$$\Lambda^{(\alpha)} := \frac{1}{4} \sum_{P \in \mathcal{P}_1} (P \otimes P^*)^{\otimes \alpha}$$

Long range magic

Long-range magic, i.e. magic that cannot be removed by short-depth quantum circuits

Tarabunga et al. PRX Quantum 4 0403 (2023)
Sewell et al. PRB 106 12513 (2022)
Frau et al. PRB 110, 045101 (2024)

SRE for mixed states

$$\tilde{M}_2(\rho) = -\log\left(\frac{\sum_{P\in\mathcal{P}_N} \mathrm{tr}(P\rho)^4}{\sum_{P\in\mathcal{P}_N} \mathrm{tr}(P\rho)^2}\right) = -\log\left(\frac{\zeta_2(\rho)}{\zeta_1(\rho)}\right), \quad \zeta_n(\rho) := \frac{1}{2^N}\sum_{P\in\mathcal{P}_N} \mathrm{tr}(P\rho)^{2n}$$

L. Leone, S. F. E. Oliviero and A. Hamma PRL 128 4 (2022)

Long range SRE

$$L(\rho_{AB}) = \tilde{M}_2(\rho_{AB}) - \tilde{M}_2(\rho_A) - \tilde{M}_2(\rho_B)$$

ZX calculus

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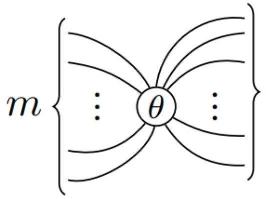
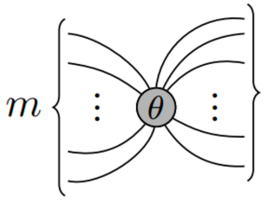
J. van de Wetering, ZX-calculus for the working quantum computer scientist (2020), arXiv:2012.13966

Useful for graphical simplifications, especially if most things are simple in Z or X basis.

Spiders

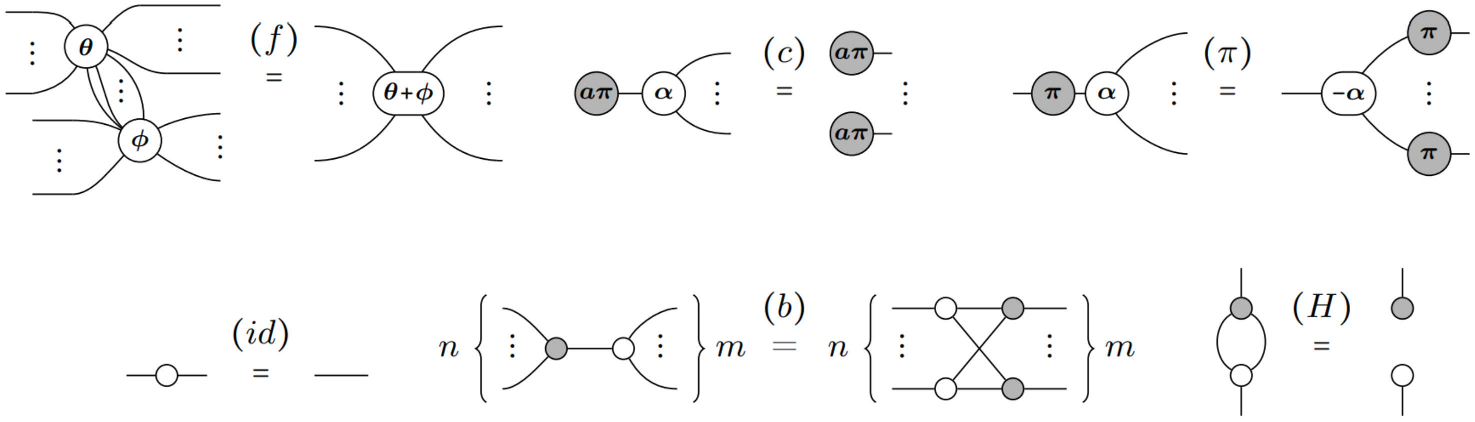


$:= |0\rangle^{\otimes m}\langle 0|^{\otimes n} + e^{i\theta}|1\rangle^{\otimes m}\langle 1|^{\otimes n}$,



$:= |+\rangle^{\otimes m}\langle +|^{\otimes n} + e^{i\theta}|-\rangle^{\otimes m}\langle -|^{\otimes n}$

Rules



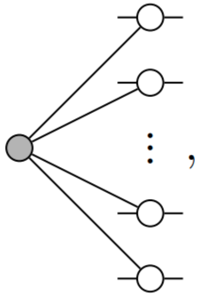
ZX calculus

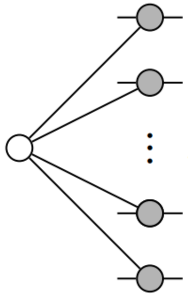
J. van de Wetering, ZX-calculus for the working quantum computer scientist (2020), arXiv:2012.13966

$$\Lambda^{(\alpha)} = \sum_{P \in \mathcal{P}_1} (P \otimes P^*)^{\otimes \alpha} = 4\Lambda_x^{(\alpha)}\Lambda_z^{(\alpha)}$$

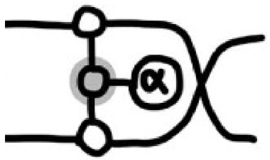
$$\Lambda_x^{(\alpha)} = \frac{1}{2}(\sigma_0^{\otimes 2\alpha} + \sigma_x^{\otimes 2\alpha}),$$

$$\Lambda_z^{(\alpha)} = \frac{1}{2}(\sigma_0^{\otimes 2\alpha} + \sigma_z^{\otimes 2\alpha}).$$

$\Lambda_z^{(\alpha)} =$  ,

$\Lambda_x^{(\alpha)} =$  ,

$$U_{\mathrm{e,o}} = \exp\left(-\mathrm{i}J_{\mathrm{e,o}}\sigma_z \otimes \sigma_z\right) \cdot \mathrm{SWAP}$$

$=$ 

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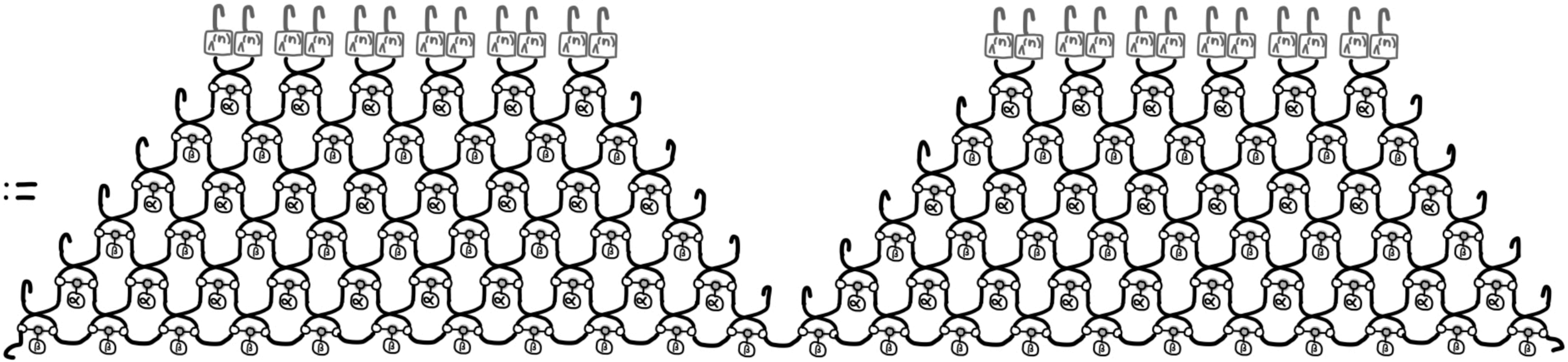
$$\zeta_n(\rho_{AB})$$

$:=$



Results

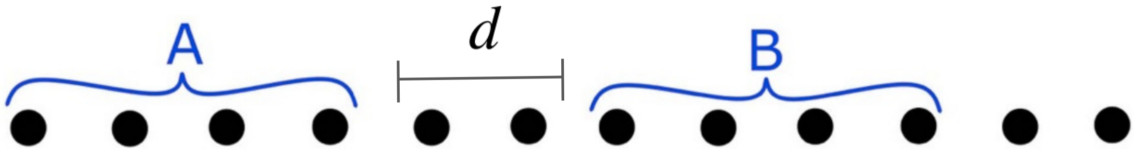
Density of magic after one layer

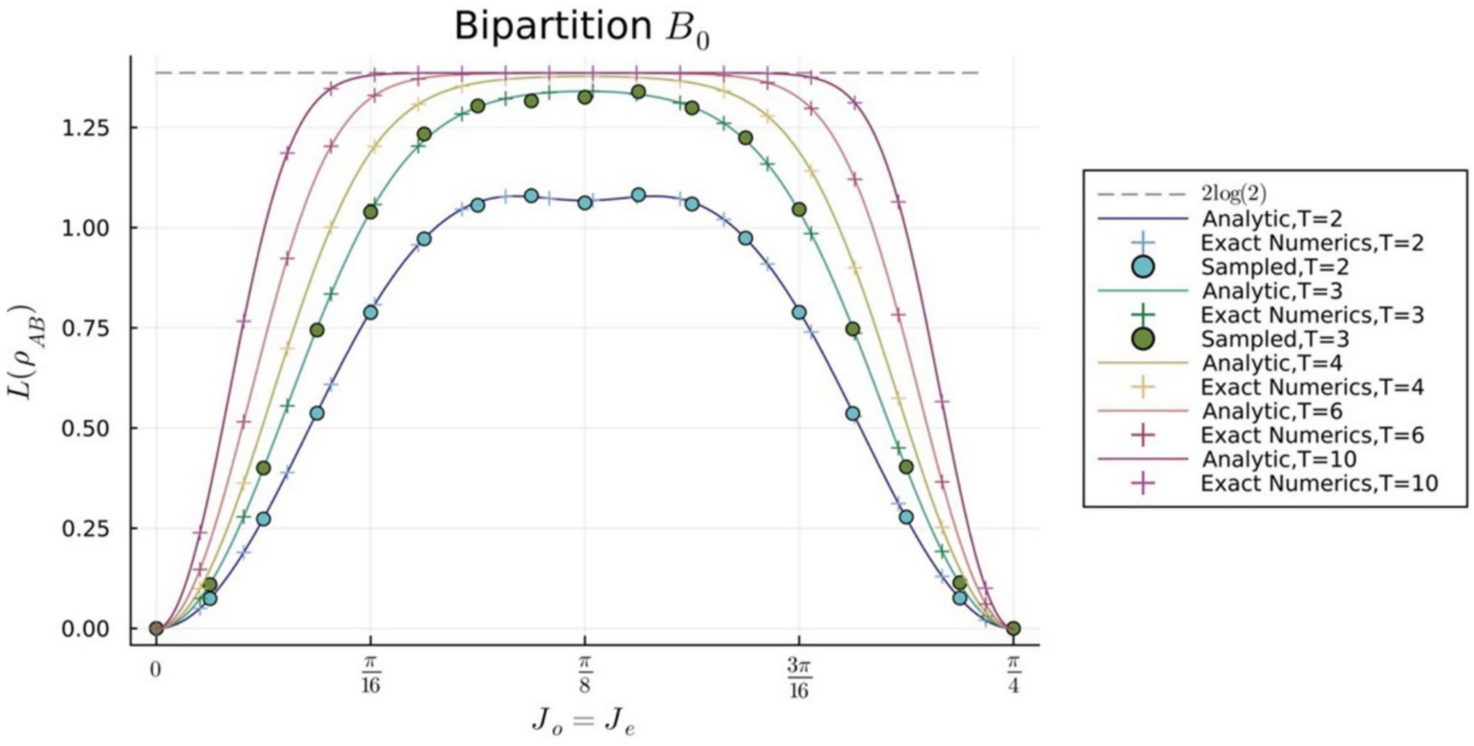
$$m_n\left(|\psi\rangle\right) = \lim_{N\to\infty} \frac{1}{N} M_n\left(|\psi\rangle\right)$$

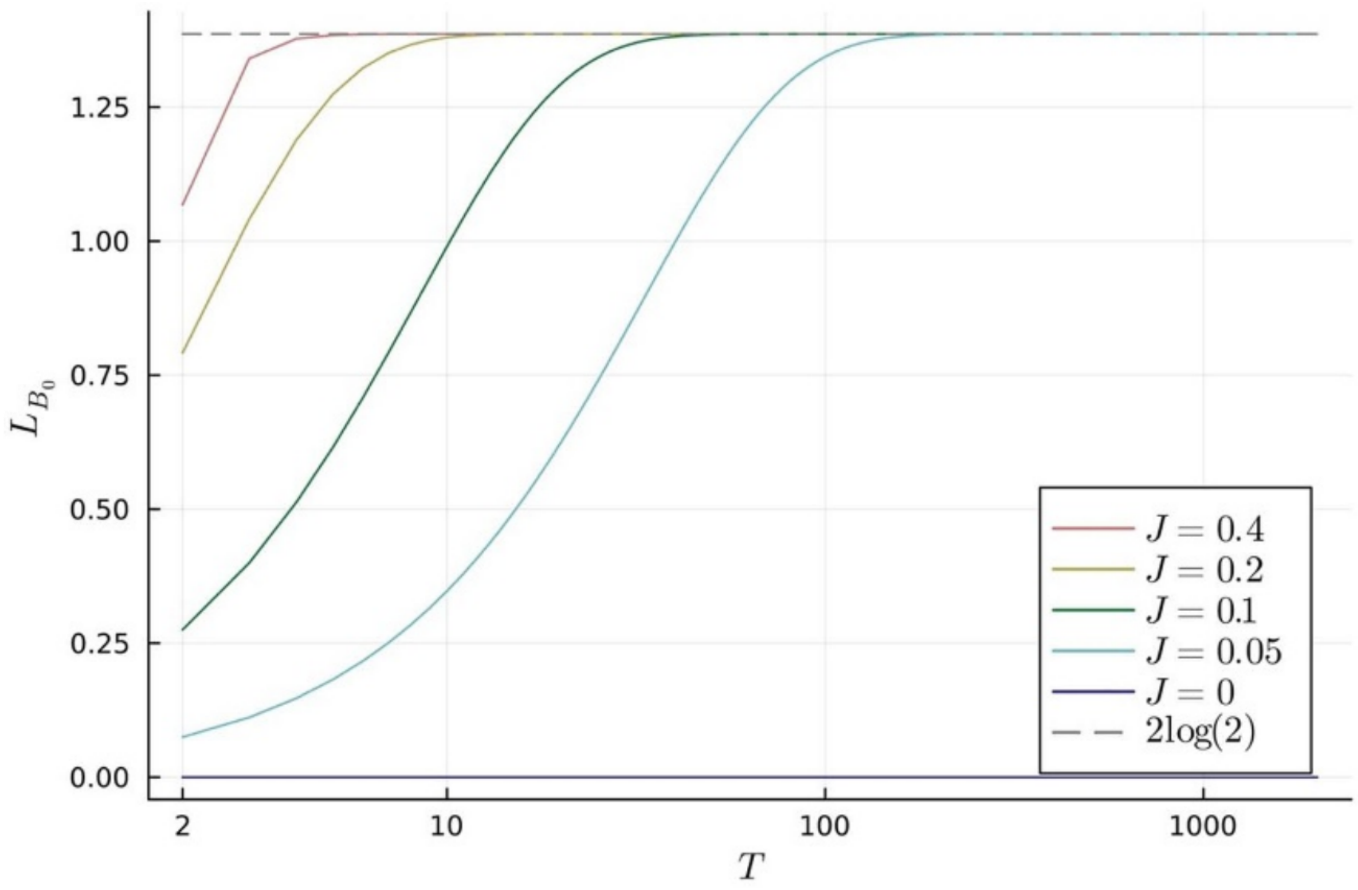
$$m_n\left(\left|\psi\left(J, 0, \frac{1}{2}\right)\right\rangle\right) = \frac{1}{2\left(1-n\right)} \log\left(\frac{1+\cos^{2n}\left(2J\right)+\sin^{2n}\left(2J\right)}{2}\right)$$

Results

Long range magic: exact results for $B_0 : N_A = N_B = 2T, d = 2(T-1), T =$ # of layers






L_{B_0}
1.25
1.00
0.75
0.50
0.25
0.00
2
10
100
1000
T
$J = 0.4$
$J = 0.2$
$J = 0.1$
$J = 0.05$
$J = 0$
$2\log(2)$

Let us change perspective from states to operators

Magic of the Heisenberg Picture

Neil Dowling, PK, Xhek Turkesh

arXiv:2408.16047

Operator magic

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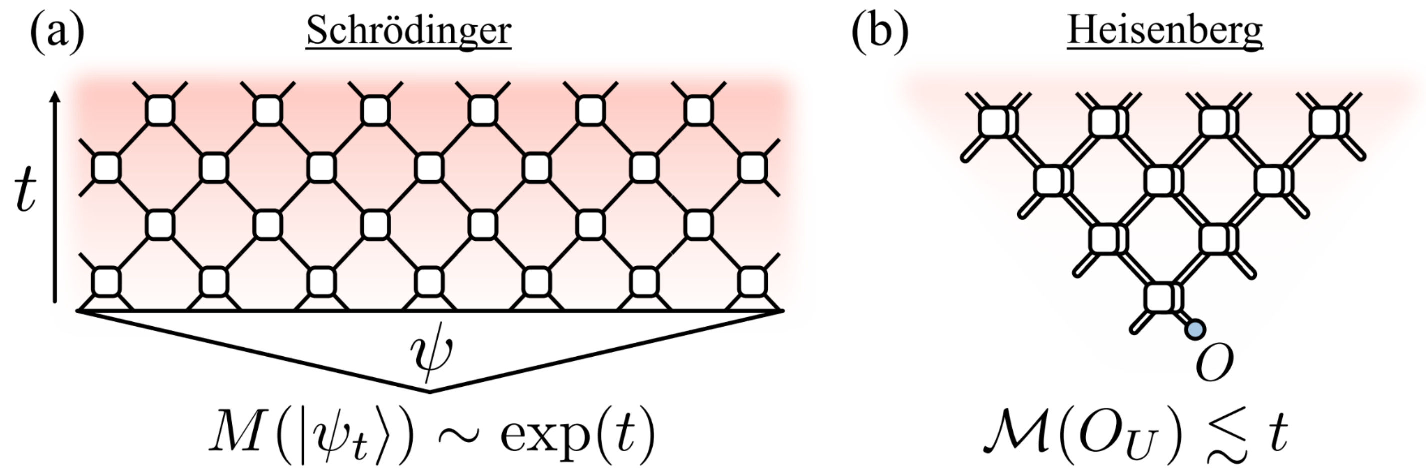
$$\langle O \rangle_t = \mathrm{tr}[O U_t \rho U_t^\dagger] \qquad\qquad O_U = U_t^\dagger O U_t$$



Operator Stabilizer Entropy (OSE)

$$\mathcal{M}^{(\alpha)}(O_U) := \frac{1}{1-\alpha}\left(\log P^{(\alpha)}(O_U) - \log P^{(\alpha)}(O)\right),$$

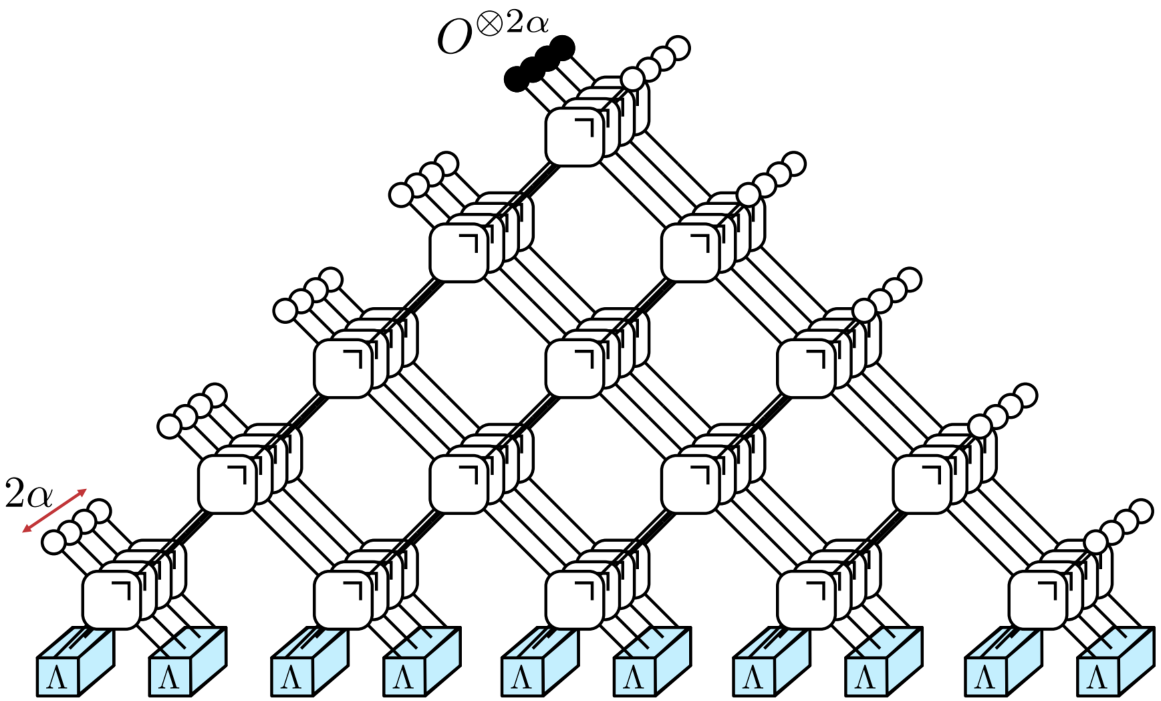
$$P^{(\alpha)}(O_U) := \sum_{P \in \mathcal{P}_N} \left(\frac{1}{D}\operatorname{tr}[O_U P]\right)^{2\alpha} \qquad D = 2^N$$

Monotone of the dynamics U

Information if we can efficiently compute expectation values
$\langle O \rangle_t = \operatorname{tr}[O U_t \rho U_t^\dagger]$ using a stabilizer formalism for any density matrix?

$$O_U \approx \underbrace{a_1 \begin{matrix} X \\ Y \\ \vdots \\ X \\ Z \end{matrix} + a_2 \begin{matrix} Z \\ X \\ \vdots \\ Y \\ Y \end{matrix} + \cdots + a_\chi \begin{matrix} X \\ Y \\ \vdots \\ Z \\ X \end{matrix}}_{\chi \sim \exp(\mathcal{M})}$$

Expressible as tensor network



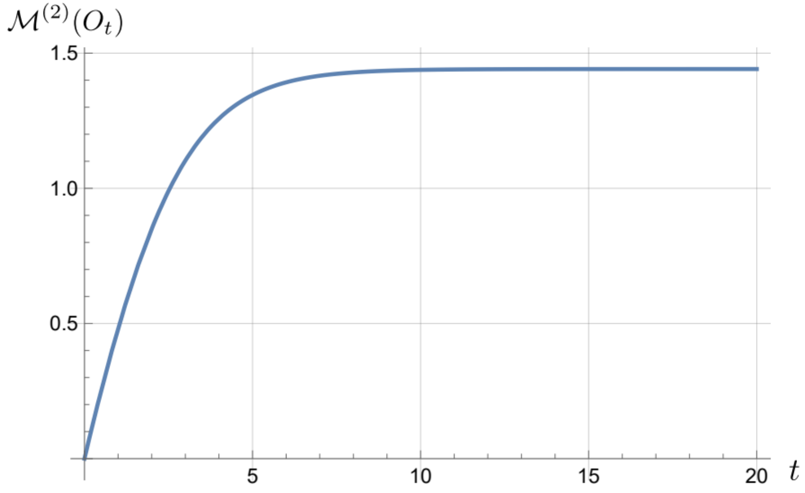
Each gate is $U \otimes U^*$

$$|\circ\rangle = \sum_i |ii\rangle$$

$$\Lambda^{(\alpha)} := \frac{1}{4} \sum_{P \in \mathcal{P}_1} (P \otimes P^*)^{\otimes \alpha}$$

Light cone!
Scales as most $\propto$ t rather than $\propto$ N

Result for OSE in DU XXZ

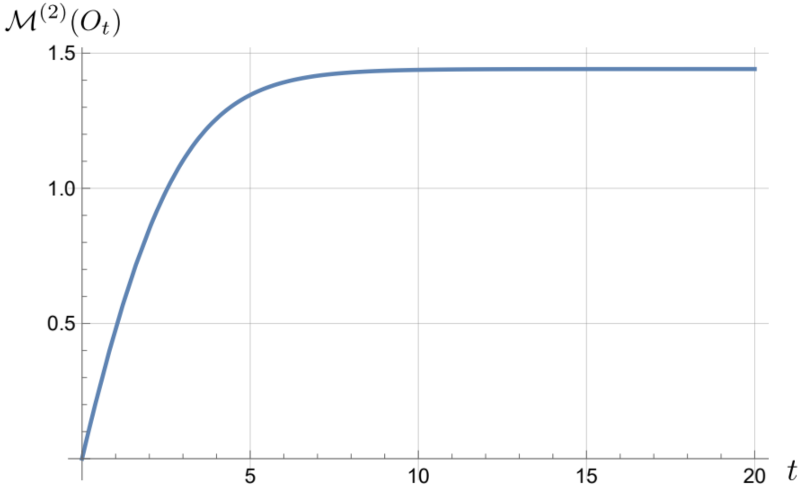


$$\mathcal{M}^{(\alpha)}(O_U) = \frac{1}{1-\alpha}\log\left(\frac{A_\alpha + \left(\cos^{2\alpha}(2J) + \sin^{2\alpha}(2J)\right)^t}{A_\alpha + 1}\right)$$

$$O = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$$

$$A_\alpha := \frac{a_z^{2\alpha}}{a_x^{2\alpha} + a_y^{2\alpha}} = \frac{(1 - a_x^2 - a_y^2)^\alpha}{a_x^{2\alpha} + a_y^{2\alpha}}$$

Result for OSE in DU XXZ



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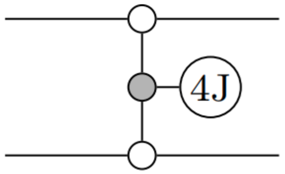
For initial operator X $\quad \mathcal{M}^{(\alpha)}(U_t^\dagger \sigma_x^{(j)} U_t) = \frac{t}{1-\alpha} \log\left(\cos^{2\alpha}(2J) + \sin^{2\alpha}(2J)\right)$

Proof

$$\mathrm{SWAP}^{\otimes N/2}\exp(-i\sum_{i\,\mathrm{even}}J\sigma_z^{(i)}\otimes\sigma_z^{(i+1)}))\sigma_x^{(j)}=\sigma_x^{(j+1)}\;\mathrm{SWAP}^{\otimes N/2}\;\exp(-iJ\sum_{i\,\mathrm{even}}(-1)^{\delta_{ji}}\sigma_z^{(i)}\otimes\sigma_z^{(i+1)}))$$

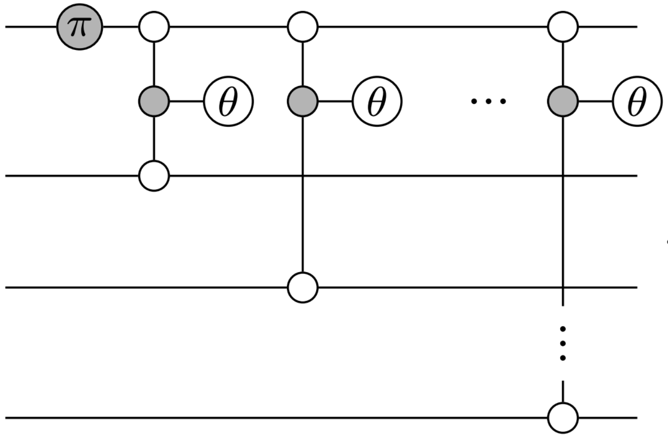
$$O_U=U_t^{\dagger}\sigma_x^{(j)}U_t=\sigma_x^{(j+t)}\exp(-i2J\sum_{i=0}^{t-1}\sigma_z^{(t+j)}\otimes\sigma_z^{(j+i)})$$

$$\exp(-i2J\sigma_z^{(a)}\otimes\sigma_z^{(b)})=$$



For initial operator X

$$O_U=$$



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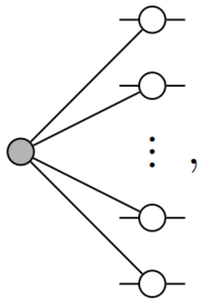
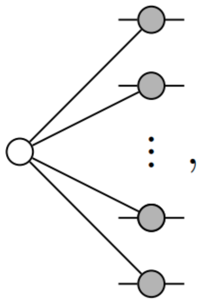
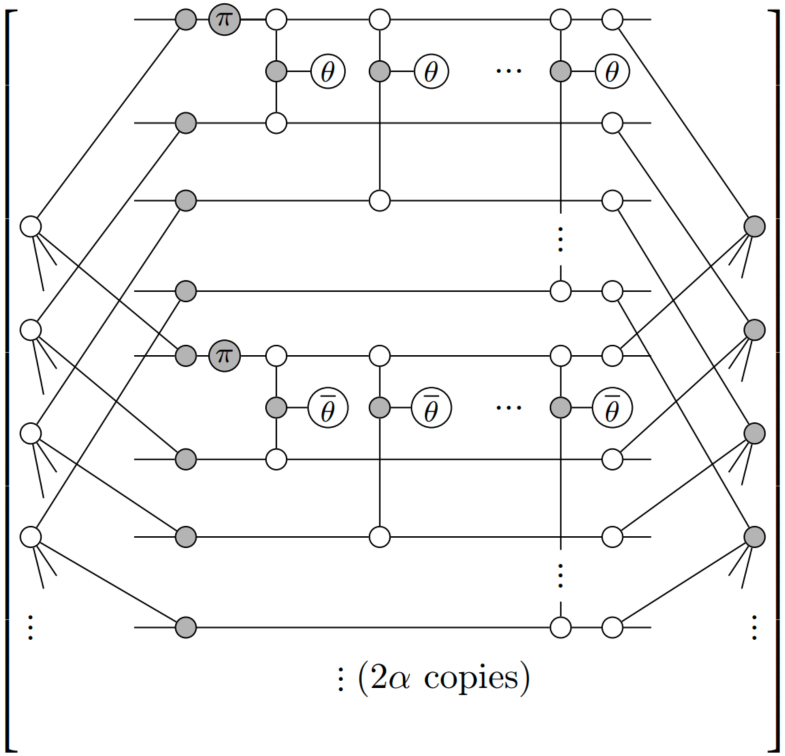
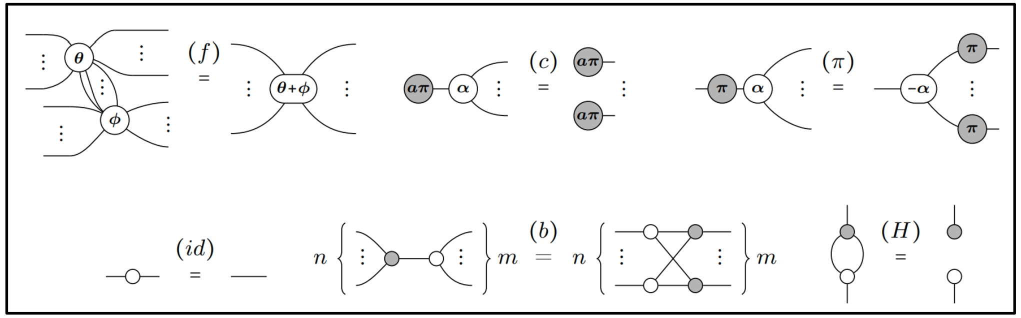
Proof

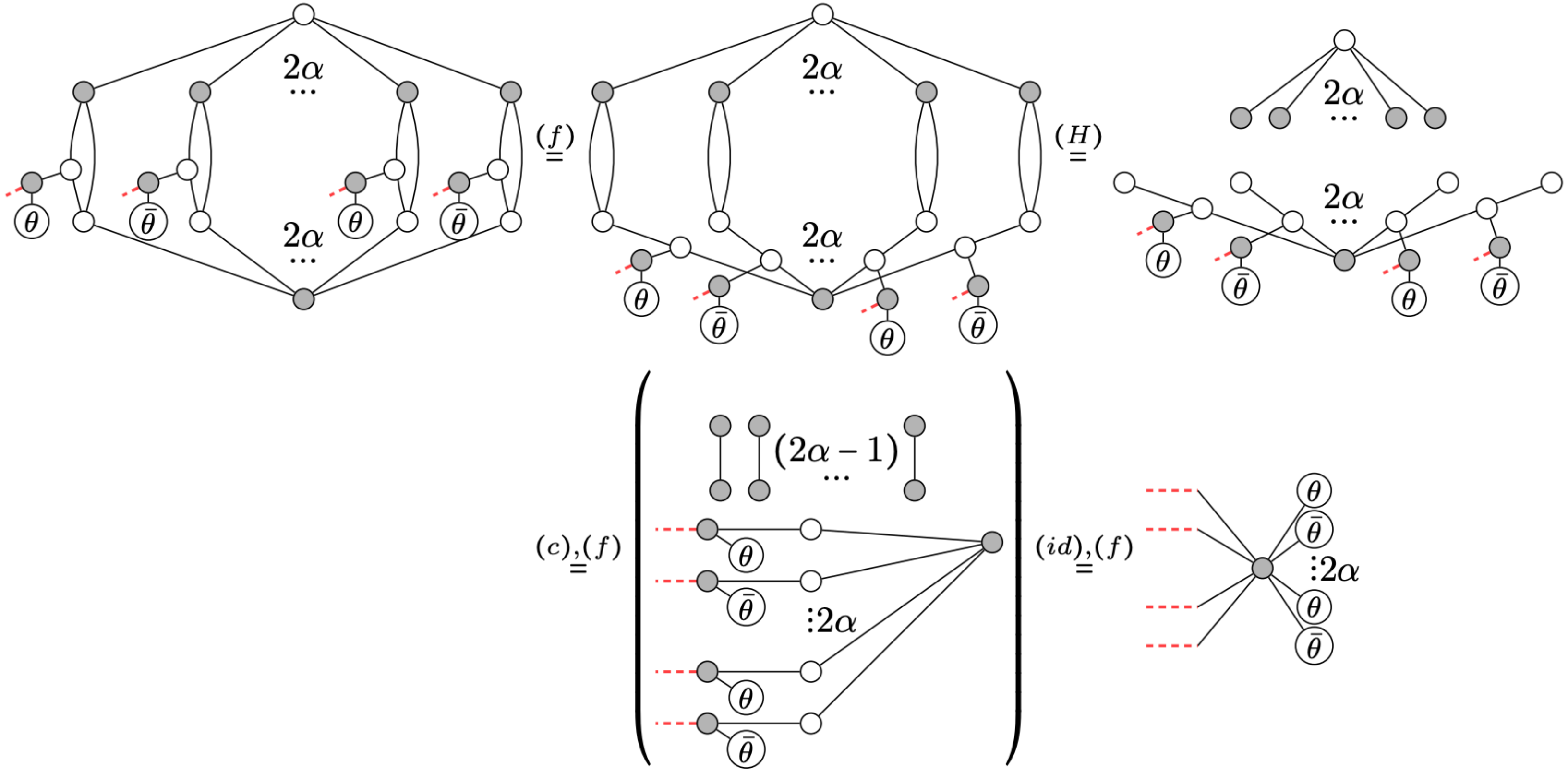
$\Lambda_z^{(\alpha)} =$  , $\Lambda_x^{(\alpha)} =$  ,

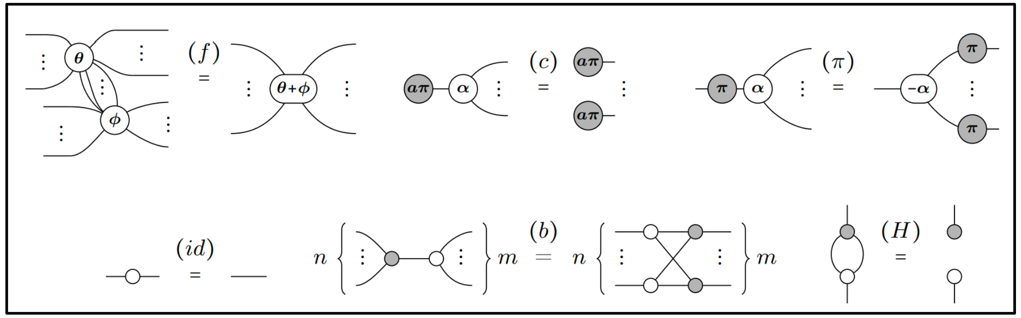
$\mathrm{tr}[(O_U \otimes O_U^*)^{\otimes\alpha}(\Lambda^{(\alpha)})^{\otimes t}] = \mathrm{tr}$ 

Proof



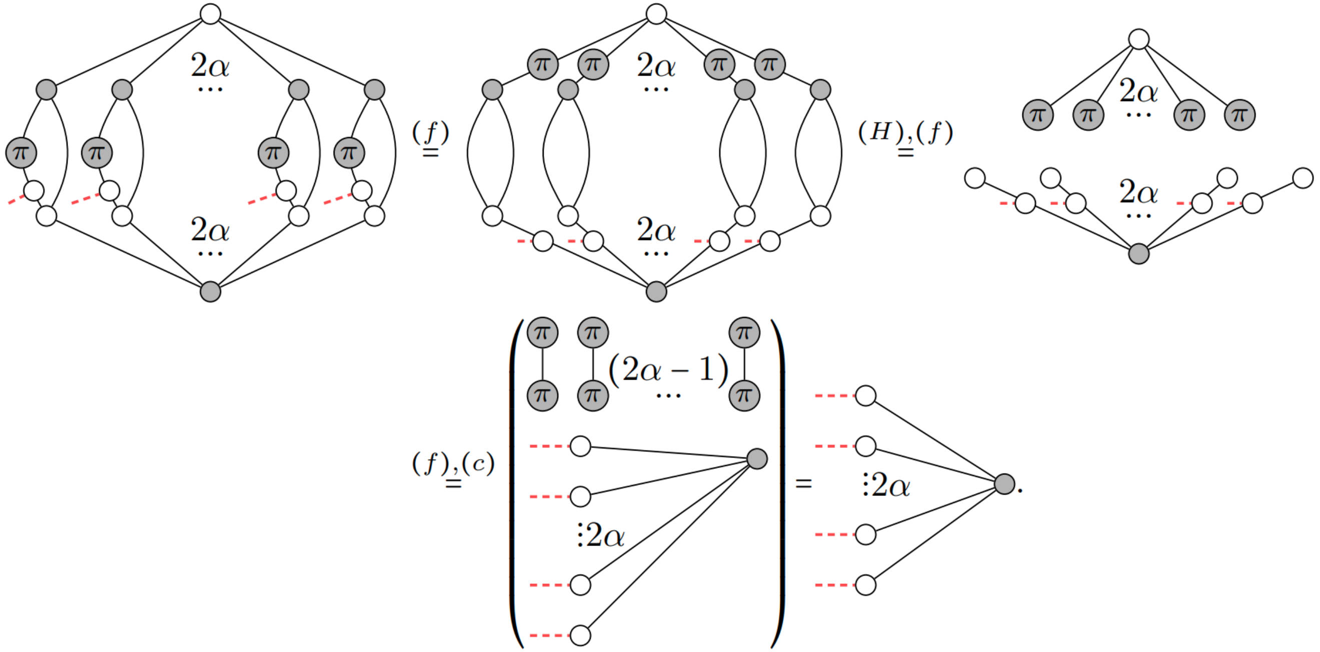
- Single site replica space

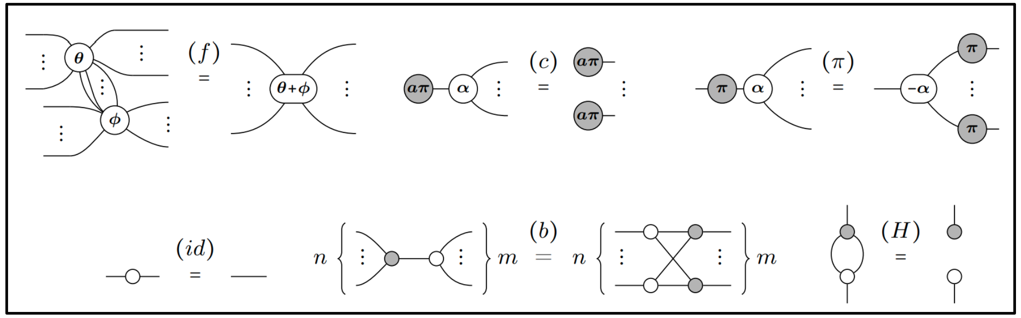




Proof

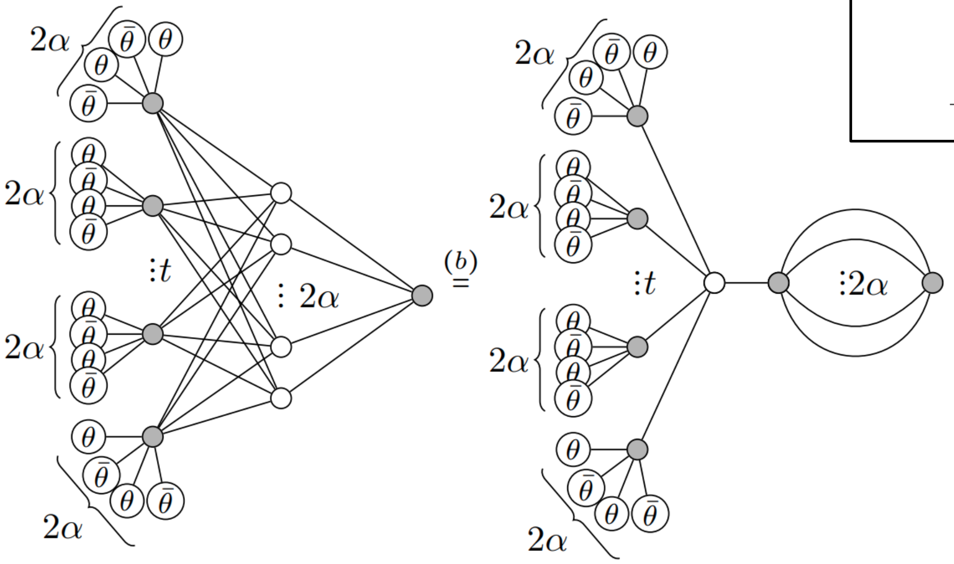
- Single site replica space with X operator

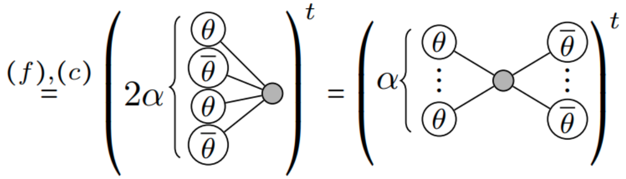




Proof

$\mathrm{tr}[(O_U \otimes O_U^*)^{\otimes\alpha}(\Lambda^{(\alpha)})^{\otimes t}] =$





$$= \left(\left(\langle 0| + \mathrm{e}^{i\theta}\langle 1|\right)^{\otimes\alpha} \left(|-\rangle\langle -|^{\otimes\alpha} + |+\rangle\langle +|^{\otimes\alpha}\right)\left(|0\rangle + \mathrm{e}^{-i\theta}|1\rangle\right)^{\otimes\alpha}\right)^t$$

$$\mathcal{M}^{(\alpha)}(U_t^\dagger\sigma_x^{(j)}U_t) = \frac{t}{1-\alpha}\log\left(\cos^{2\alpha}(2J)+\sin^{2\alpha}(2J)\right)$$

This is for X, for general operator it is similar

Summary and conclusions

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Magic generation in MB: quick growth to finite density

Long-range magic saturates to maximal value

Analytical results are possible, DU XXZ using ZX calculus

Jordi Arnau Montañà López, PK
J. Phys. A: Math. Theor. 57 475301 (2024)

Operator stabilizer entropy (OSE)

Neil Dowling, PK, Xhek Turkesh
arXiv:2408.16047